

The Infinitesimal Slice: Energy Equilibrium of a Hydrostatic Ideal Gas and the Adiabatic Lapse Rate

Joseph E. Postma

joepostma@live.ca

Abstract

A hydrostatic ideal gas in a gravitational field admits two closed energy accounts of the vertical temperature structure. One maximises entropy at fixed molecular energy in the presence of a heat bath and yields an isothermal column with the barometric exponential; the other requires that no heat be added to a unit mass as it changes height in the hydrostatic pressure field and yields the dry adiabatic lapse rate $\frac{dT}{dh} = -g/c_p$. These two accounts cannot both be the equilibrium of the same gas. The present paper derives the second of them from conservation of mass, momentum, and energy on an infinitesimal horizontal slice, and does so without taking bulk convective transport as a premise: mass conservation constrains the two thermal streams through the slice, momentum conservation is hydrostatic balance, and energy conservation, once the work against the hydrostatic pressure field is recognised as belonging to the unit mass, is the enthalpy statement $c_p dT + g dh = 0$. The same equation is recovered from the First Law of a reversible adiabatic parcel together with hydrostatic balance, and that identity is the consistency of one physical closure rather than two independent miracles. The isothermal solution is the equilibrium of a thermostatted box, not of an unguarded hydrostatic atmosphere. Nocturnal inversions and the stratosphere are layers in which heat is added or removed, and so they confirm the closure rather than refute it. The tropospheric profile occupies the adiabatic energy equilibrium. A formal exponential pressure law appears only if temperature is imposed as a constant, which is a state the energy moment of the slice does not grant. The argument is intended for advanced undergraduate and graduate thermal physics, as a classroom alternative to the labelled-parcel derivation of the dry adiabat.

Keywords: dry adiabatic lapse rate; hydrostatic equilibrium; enthalpy; mutual exclusivity; Maxwell-Boltzmann barometer; infinitesimal slice; physics education

1. Introduction

The troposphere decreases in temperature with height, and the observed global-mean environmental lapse rate is of order 6.5 K km^{-1} , lying between the dry adiabatic value $\Gamma_a = g/c_p \approx 9.8 \text{ K km}^{-1}$ and the slower moist rate produced when latent heat offsets expansional cooling [1-3]. Two first-principles routes to the dry value have long been standard: one writes the First Law for a reversible adiabatic parcel and eliminates pressure with hydrostatic balance, while the other writes the specific enthalpy plus gravitational potential of a fixed mass and sets the differential to zero. Both give

$\frac{dT}{dh} = -g/c_p$, and no free parameter is adjusted to force the match, the constants being simply laboratory thermodynamics and terrestrial gravity.

A rival closed account of the same column is older still. In an external gravitational potential the canonical distribution of a classical gas factorises in position and momentum, yielding a single temperature and the barometric law $n \propto \exp(-m g h / k T)$ [4,5]. Maxwell and Boltzmann took this isothermal hydrostatic state to be the thermal equilibrium of a gas in gravity, whereas Loschmidt argued that the bottom of the column must be hotter. The historical exchange is summarised by Brush [9]. The present paper does not rest on that history but restates the two closures in modern thermodynamic language. Modern kinetic theory sides with the isothermal solution for a gas coupled to a heat bath, while modern meteorology sides with the adiabat as the convective reference state of the troposphere, and both cannot be the energy equilibrium of the same unguarded hydrostatic mass. That is a problem of mutual exclusivity [6], not a problem of insufficient algebra.

The present paper has three aims. First, it organises mass, momentum, and energy on a single infinitesimal horizontal slice, so that the dry adiabat is seen as the energy moment of the same construction that supplies hydrostatic balance. Second, it states the two energy closures with their conserved quantities written explicitly, and it identifies the heat bath as the hidden assumption that selects the isothermal solution. Third, it answers the principal objections: that $dq = 0$ is an extra assumption rather than the definition of the unguarded system; that molecular streams through a plane of rest cannot perform expansion work; that identifying a characteristic speed with enthalpy inserts g/c_p by hand; that the absence of bulk vertical motion forbids c_p ; that laboratory cylinders have never been seen to become isothermal, or alternatively that the atmosphere itself, by inversions and a warming stratosphere, refutes a mandatory lapse.

The paper is written for advanced undergraduate and beginning graduate courses in thermal physics, statistical mechanics, and atmospheric physics, and for instructors who teach the hydrostatic atmosphere from those courses. Students ordinarily meet the dry adiabat as a labelled parcel that rises without exchanging heat. The learning objectives are to distinguish the isothermal barometer from the dry adiabat as two complete energy accounts of one gradient; to derive $dT/dh = -g/c_p$ from mass, momentum, and energy on a plane of vanishing net mass flux, without taking a convective cell as a premise; and to see that a one-way complete parcel through a plane of rest would leave a hole or a net flux, so that equal and opposite streams are required by mass conservation. The slice construction can replace or follow the parcel derivation in a lecture or problem class that already uses the First Law and hydrostatic balance [1-3,8].

The conclusion is narrow and physical. Hydrostatics plus the ideal-gas law do not by themselves pick $T(h)$, and so an energy closure is required. For a collisional hydrostatic ideal gas that is not coupled to a thermostat, that closure is that no heat is added to a unit mass as it changes height, and the lapse rate $-g/c_p$ is then mandatory. The relation does not mention heating from below because it is the energy law of that system, not a parameterisation of a thunderstorm: heating from below decides whether radiation is allowed to impose some other slope, but it does not write g or c_p . Bulk

convection is how a deep, heated column realises the energy equilibrium; it is not a second cause of the slope.

Section 2 states the physical necessities of the derivation; Section 3 constructs the slice and writes mass and momentum; Section 4 writes the energy moment and recovers the dry adiabat, together with the equivalent parcel derivation; Section 5 states the mutual exclusivity of the two closures; Section 6 integrates the hydrostatic structure and states what the reference pair (T_0, P_0) is and is not; Section 7 includes latent heat; Section 8 treats objections; Section 9 states the empirical content; and Section 10 concludes.

2. Physical setting and necessities

Consider a gravitational field of locally constant acceleration g (positive), directed opposite to increasing geometric height h , and a collisional ideal gas occupying that field. The following conditions are not decorative: each is necessary for the derivation that follows, and each is physically realised in the terrestrial troposphere.

- N1. **Continuum.** The molecular mean free path λ is much smaller than the pressure scale height $H = R T / g$. For air near the surface, λ is of order $0.1 \mu\text{m}$ and H is of order 8 km , so the gas is a fluid. Ballistic, collisionless kinetic theory is not the model, and conclusions drawn from a collisionless Maxwellian in gravity are not transferable without argument.
- N2. **Hydrostatic balance.** The bulk vertical acceleration is negligible compared with g . The weight of each layer is borne by the pressure difference across it. This is the observed mechanical state of the large-scale troposphere [2].
- N3. **Ideal gas.** Locally, $P = \rho R T$, with $R = k/m$ the specific gas constant. Dry air is adequately ideal in the troposphere. The specific heats c_v and $c_p = c_v + R$ are treated as constant.
- N4. **No heat bath.** The column is not in thermal contact with walls that own a single temperature. A laboratory cylinder a metre high has $m g h / k T \sim 10^{-4}$ and conducting walls, and those walls impose one temperature. An atmosphere has no such walls. Sunshine maintains the existence of the gas against freeze-out; it is not, in this derivation, a boundary condition that fixes T at one value.
- N5. **Zero bulk vertical mass flux, not zero molecular flux.** Mechanical equilibrium means that the net mass flux through a horizontal surface vanishes, which is not to say that molecules do not cross the slice. They do, at thermal speed, in equal and opposite streams, and the energy accounting is performed on those streams.

If any of N1-N5 is dropped, a different problem is being solved: dropping N1 returns collisionless kinetic theory, dropping N2 returns non-hydrostatic dynamics, dropping N3 returns a general fluid, dropping N4 returns the thermostatted box whose equilibrium is isothermal, and dropping N5 confuses vanishing mean wind with vanishing thermal motion. The derivation below keeps all five.

3. The infinitesimal slice: mass and momentum

3.1 Geometry

Take a horizontal mathematical surface of unit area at height h , and a neighbouring surface at $h + dh$. The slab between them is the slice. Label the density and velocity of the upward-moving population that crosses the surface by $(\rho^\uparrow, v^\uparrow)$ and those of the downward-moving population by $(\rho^\downarrow, v^\downarrow)$. These are the two thermal streams, and they exist because the gas has a Maxwellian, or locally near-Maxwellian, velocity distribution; they are not a parameterisation of a convective cell (Figure 1). Both streams share the same characteristic speed v_0 at the reference height because they are the two vertical hemispheres of one local Maxwellian. The Maxwellian is three-dimensional: v_0 is the characteristic vertical speed of each hemisphere.

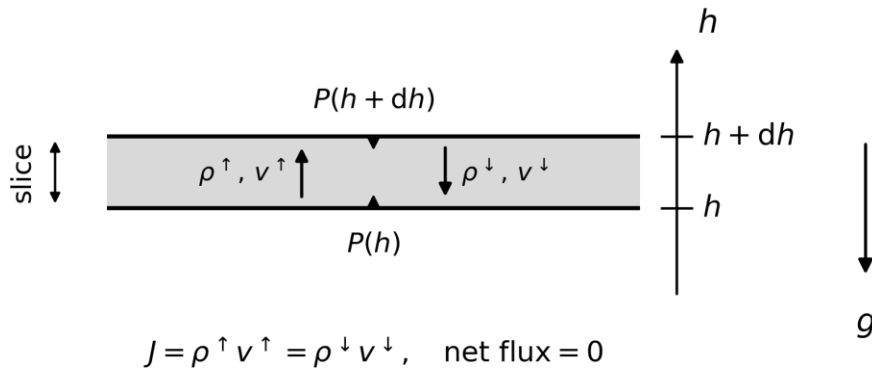


Figure 1. The infinitesimal slice. Equal one-way mass fluxes $J = \rho^\uparrow v^\uparrow = \rho^\downarrow v^\downarrow$ occupy the slab between h and $h + dh$ at once. Gravity acts downward. The pressure on the two faces is the vertical momentum flux through the slice. The net mass flux vanishes.

3.2 Mass conservation

In mechanical equilibrium the net mass flux through the surface vanishes. Equal mass therefore crosses in each direction per unit time, and that common one-way flux is written J :

$$\rho^\uparrow v^\uparrow = \rho^\downarrow v^\downarrow \equiv J \quad (1)$$

Here J is the one-way mass flux through the slice, that is, mass per unit area per unit time carried by either stream. The net flux is then $J - J = 0$. Equation (1) is the zeroth moment of the slice: it is necessary, but it is not sufficient for $P(h)$ or $T(h)$.

Over the time dt in which a stream crosses a distance of order dh ,

$$dh = v_0 dt \quad (2)$$

so $dt = dh / v_0$, where v_0 is that common characteristic speed before the gravitational increment of that interval. Gravity distinguishes the two speeds:

$$v^{\uparrow} = v_0 - g dt = v_0 - \frac{g dh}{v_0} \quad (3)$$

$$v^{\downarrow} = v_0 + \frac{g dh}{v_0} \quad (4)$$

Substitution into (1) gives

$$\frac{\rho^{\uparrow}}{\rho^{\downarrow}} = \frac{v_0^2 + g dh}{v_0^2 - g dh} \quad (5)$$

The right-hand side is greater than one, so the upward stream, fed from below, is denser and the downward stream, fed from above, is thinner, and density therefore decreases with height. That is as far as mass conservation can go. One might be tempted to identify the two streams with the gas below and the gas above, read (5) as $\rho(h)/\rho(h + dh)$, and expand for small dh to obtain $\frac{1}{\rho} \frac{d\rho}{dh} = -2g/v_0^2$. The factor of two is only $(1 - x)/(1 + x) \approx 1 - 2x$: gravity was applied to both streams, so their speed difference is $2g dt$. Setting $v_0^2 = RT$ would then give a scale height $RT/2g$, half of the hydrostatic value $H = RT/g$. That halved scale height is the result of treating a mass-conservation ratio of two streams at one height as if it were the bulk density gradient with height. The latter is fixed by the weight of the column, which is a statement of momentum, not of mass flux. The ratio $\rho^{\uparrow}/\rho^{\downarrow}$ is a contrast at one height, between the two directions crossing the same plane; $\rho(h)/\rho(h + dh)$ is a contrast between two heights, which is the weight of the column. Those are not the same object, and the latter is why the next section turns to momentum.

3.3 Momentum conservation: hydrostatic balance

Pressure on the slice is the vertical momentum flux through it. Gravity removes vertical momentum from the mass inside the slice at the rate $\rho g dh$. In a stationary state the pressure difference supplies exactly that:

$$P(h) - P(h + dh) = \rho g dh \quad (6)$$

$$\frac{dP}{dh} = -\rho g \quad (7)$$

Equation (7) is the continuum statement of gravity acting on the vertical momentum of the two streams, that is, hydrostatic balance [1,2]. Combined with the ideal-gas law

$$P = \rho RT \quad (8)$$

it becomes

$$\frac{dP}{P} = -\frac{g dh}{RT} \quad (9)$$

Equation (9) still contains $T(h)$. Hydrostatics plus the ideal-gas law do not determine the temperature structure, and so an energy statement is required. If temperature is imposed constant by hand, (9) integrates at once to the barometric exponential

$$P(h) = P_0 \exp\left(-\frac{g h}{R T_0}\right) \quad (10)$$

Equation (10) is algebra under an imposed assumption, but nothing in (1)-(9) has permitted holding T as fixed. The names Maxwell-Boltzmann equilibrium and Laplace-Perrin barometer attach to that imposed case [4,5,7], and they describe a thermostatted gas, not the unguarded hydrostatic atmosphere of Section 2. The energy moment of the same slice, developed next, does not permit the constant temperature of (10) as a physical state of that atmosphere.

4. Energy of the streams

4.1 The First Law at locally defined pressure

A unit mass in the slice sits in a pressure field already fixed by (7). Its specific volume is $\alpha = 1/\rho = R T / P$. The First Law per unit mass for a reversible process in an ideal gas may be written in the enthalpy form [3,8]

$$dq = c_p dT - \alpha dP \quad (11)$$

or, equivalently,

$$dq = c_p dT - dP / \rho \quad (12)$$

The specific enthalpy is $c_p T = c_v T + P/\rho = c_v T + R T$. The extra $R T = P/\rho$ is not a second pile of heat stored in the molecules, but the work the unit mass can do, or has done on it, against the surrounding pressure, and that work belongs in the energy budget of any mass whose pressure changes. In a hydrostatic atmosphere the pressure of a unit mass changes whenever its height changes, whether the change is a coherent parcel displacement or a thermal crossing of the slice. Using c_v alone would describe a rigid box, in which the mass cannot change volume against a neighbouring pressure, and air is not a rigid box. The fingerprint of c_p is $P dV$, and $P dV$ is already present in hydrostatics.

4.2 Substitution of hydrostatics

Equation (7) is $dP = -\rho g dh$. Substitution into (12) yields

$$dq = c_p dT + g dh \quad (13)$$

This is the energy moment of the slice. Gravity appears because hydrostatic pressure changes with height, and enthalpy appears because the unit mass is at a defined pressure. No bulk convective cell has been introduced, and no heating from below has been introduced: the equation is simply the First Law of a hydrostatic ideal gas, written per unit mass.

4.3 The meaning of $dq = 0$

Equation (13) still contains dq . Two closures present themselves.

Closure A. The unit mass is in contact with a heat bath, or entropy is maximised at fixed molecular energy $E = \int \rho(c_v T + g h) dV$ inside a box whose walls own the temperature. Then dq is whatever is required to keep a single $\beta = 1/k T$. The result is isothermal (Section 5).

Closure B. No heat is added to the unit mass as it changes height. Then

$$dq = 0 \quad (14)$$

and (13) becomes

$$c_p dT + g dh = 0 \quad (15)$$

$$\frac{dT}{dh} = -\frac{g}{c_p} \quad (16)$$

Closure B is not an extra dynamical assumption piled onto hydrostatics after the fact; it is the definition of the unguarded system of Section 2, namely that there is no thermostat. Sunshine maintains the column against freeze-out, but it does not assign a single T to every height. A unit mass that crosses the slice exchanges gravitational potential with enthalpy and with nothing else, and that is the physical content of (14). A counter-claim that (14) was “assumed, therefore the lapse is not derived” confuses a constitutive statement about the system with a free parameter. The rival constitutive statement is the heat bath, and those two statements cannot both be true of the same mass (Section 5).

4.4 The dry adiabat

Integration of (16) at constant g and c_p gives the linear temperature law

$$T(h) = T_0 - \left(\frac{g}{c_p}\right) h \quad (17)$$

With $g \approx 9.81 \text{ m s}^{-2}$ and $c_p \approx 1004 \text{ J kg}^{-1} \text{ K}^{-1}$ for dry air,

$$\Gamma_d \equiv g/c_p \approx 9.76 \text{ K km}^{-1} \quad (18)$$

This is the dry adiabatic lapse rate [1-3,8], a direct theoretical result whose constants are not tuned to a target gradient.

In integrated form, for a mass m ,

$$U = m c_p T + m g h \quad (19)$$

$$dU = 0 = m c_p dT + m g dh \quad (20)$$

Equation (19) is the same statement as (15). The slice says it about the two streams: upward mass arrives at $h + dh$ having converted $g dh$ of enthalpy into potential energy, so it is cooler by

$(g/c_p) dh$; downward mass does the reverse. Equal mass fluxes (1) imply no net mass transport and a temperature jump $2(g/c_p) dh$ between the two streams. The factor of two is a contrast of equal and opposite changes, twice the environmental difference between the faces of the slice, and the environmental gradient remains (16). It is the temperature analogue of the two-stream ratio (5), which is likewise not to be read as a bulk gradient over dh .

4.5 Equivalence with the parcel derivation

The textbook parcel route [1-3] begins with (11) at $dq = 0$ and substitutes $dP = -\rho g dh$. The result is again (16). The parcel treatment says “adiabatic” out loud and the slice says “no heat is added,” which is the same closure applied to the same hydrostatic pressure field. That is consistency of physics, not two independent proofs that convection is irrelevant, and not a circular insertion of g/c_p . If c_v were used in place of c_p , both routes would give $-g/c_v \approx 14 K km^{-1}$, which air does not do. The need for c_p is the fingerprint of expansion work against hydrostatic pressure, present in both treatments because both treatments describe a compressible mass, not a rigid box.

A characteristic-speed identification $\frac{1}{2}v_0^2 = c_p T$ recovers (15) from the velocity update (3)-(4), because $v dv = -g dh$. That identification is a bridge between the stream picture and enthalpy, not an independent derivation: the derivation is (11)-(16).

5. Mutual exclusivity of the two energy closures

5.1 Statement

Let the phenomenon be the vertical temperature structure $T(h)$ of a hydrostatic ideal gas, and let the exclusivity variable Φ be the gradient dT/dh . The formal schema of additive concurrent mechanisms [6] is applied here to the two energy closures of that column. In that schema the exclusivity variable was called V ; it is written Φ here so that it is not confused with thermodynamic volume. Two complete energy accounts are on offer.

E₁ (unguarded hydrostatic mass). Conserved quantity per unit mass: $c_p T + g h$. Closure: $dq = 0$. Result: $\frac{dT}{dh} = -g/c_p$.

E₂ (thermostatted column). Conserved quantity of the closed volume: $E = \int \rho(c_v T + g h) dV$. Closure: maximum entropy at one β , or coupling to a heat bath. Result: T independent of h , and $n \propto \exp(-m g h / k T)$.

These mechanisms satisfy the mutual-exclusivity constraints of [6]: they are additive on the same Φ if both are declared fully real; they are independent (enthalpy of an unguarded mass versus molecular energy in a thermostat); they target the same gradient; they would, if concurrent, both claim to be the equilibrium; they cannot hide behind a compensatory factor that has been independently measured. If E_1 already fixes Φ , then E_2 cannot contribute a non-zero full effect to Φ . Conversely, if E_2 is the equilibrium, E_1 is not, and they cannot both run.

Closure	Equilibrium $T(h)$
Heat bath, maxent at fixed $c_v T + g h$ (E2)	$T = T_0$ (isothermal)
No heat added, hydrostatic mass (E1)	$\frac{dT}{dh} = -g/c_p$

5.2 The heat bath is the hidden assumption

The Boltzmann distribution in an external potential,

$$f(p, h) \propto \exp\left(-\beta\left(\frac{p^2}{2m} + m g h\right)\right) \quad (21)$$

is the collisional equilibrium of the canonical ensemble [4,5]. Gravity is in the Hamiltonian, and collisions drive toward (21); they do not replace c_v with c_p . Position and momentum factorise, and so there is one temperature. This is not in dispute as a theorem about a thermostatted gas; it is in dispute as a description of the unguarded hydrostatic atmosphere.

Equation (21) smuggles the heat bath in the symbol β . A single β is owned by the walls, or by the statement that the entire column shares one kinetic temperature, and an atmosphere has no such walls. Assigning (21) to “the equilibrium of a real gas in gravity” is therefore the assignment of the thermostatted box to a system that is not a thermostatted box, and that is the false step. Mutual exclusivity then does the rest: if the bath is present, the lapse is not the equilibrium, and if the bath is absent, (21) is not the equilibrium.

5.3 Collisionless kinetic theory

A collisionless Maxwellian in gravity is isothermal, with $n \propto \exp(-m g h / k T)$. A collisionless stack of independent orbits can hold a non-uniform $T(h)$ frozen by the initial amplitudes, while mixing collisions in a thermostat wash the latter out toward (21). It is therefore not legitimate to discard the isothermal solution on the pretext that “isothermal came from collisionless kinetic theory.” Isothermal is the thermostatted collisional result, with which the collisionless Maxwellian happens to agree, and neither is the unguarded hydrostatic energy equilibrium of Section 4.

The continuum condition N1 already excludes the collisionless model as the model of air: mean free path much less than scale height means the gas is a fluid at defined pressure, and the energy of a fluid mass at defined pressure is enthalpy. That is why c_p enters, and why it does not enter by matching the weather after the fact.

5.4 Zero mean motion is not vanishing process

Section 2 requires vanishing bulk vertical mass flux, while Section 4 treats the streams as exchanging enthalpy with potential energy, and there is no contradiction. Equation (1) says the two fluxes are equal; it does not say they are zero. Molecules cross the slice at thermal speed, and each

crossing changes $g h$. Because P differs by $dP = -\rho g dh$ between neighbouring heights, each crossing also changes specific volume. That is expansion work on the streams, not a requirement that a balloon of air be identified and marched through a thunderstorm. Hydrostatics itself is built from the momentum of those same crossings, and nobody discards (7) because the free path is short. The energy moment of the same crossings is (13), after which the next collision rebuilds a local Maxwellian at the new enthalpy. That is a fluid, not a refutation of c_p .

6. The hydrostatic structure once $T(h)$ is known

With (17) in hand, (9) becomes

$$\frac{dP}{P} = -\frac{g dh}{R (T_0 - \Gamma_d h)} \quad (22)$$

Integration yields Poisson's relation for an isentropic ideal gas [3,8],

$$\ln\left(\frac{P}{P_0}\right) = \frac{c_p}{R} \ln\left(\frac{T}{T_0}\right) \quad (23)$$

The same pairing, solved for temperature at a given pressure, is

$$T = T_0 \left(\frac{P}{P_0}\right)^{\frac{R}{c_p}} \quad (24)$$

Equation (24) is (23) written in the pressure coordinate. It follows at once from dividing the energy law (15) by the hydrostatic law (7), which eliminates both dh and g , so that $T(P)$ does not contain gravity. That cancellation is real, but it does not make the atmosphere independent of g : the height profile (17) still falls as g/c_p . Equation (25) is the same statement as (24), solved for pressure:

$$P(h) = P_0 \left(\frac{T(h)}{T_0}\right)^{\frac{c_p}{R}} \quad (25)$$

$$P(h) = P_0 \left(1 - \frac{g h}{c_p T_0}\right)^{\frac{c_p}{R}} \quad (26)$$

For air, $c_p/R = \gamma/(\gamma - 1) \approx 3.5$. Equation (26) is the exact companion of the linear temperature (17). It is not an exponential, though it becomes one to first order when $g h \ll c_p T_0$:

$$\left(1 - \frac{g h}{c_p T_0}\right)^{\frac{c_p}{R}} \approx \exp\left(-\frac{g h}{R T_0}\right) \quad (27)$$

The specific heat is $c_p \approx 1004 \text{ J kg}^{-1} \text{ K}^{-1}$, so $c_p T_0 / g = T_0 / \Gamma_d$ is about 30 km, which is the formal height at which the dry adiabat (17) would reach absolute zero. The condition $g h \ll c_p T_0$ is therefore h much less than that scale. Over one kilometre, $g h / (c_p T_0)$ is only a few percent; over

the depth of the troposphere it is of order one third, and (27) is then only a local approximation to (26).

The exponential (10) and the linear law (17) are therefore the two natural integrals of the same hydrostatic relation (9), under the two rival closures of Section 5. Only (17) with (26) is the unguarded energy equilibrium; equation (10) is what (9) would say if temperature were prevented from adjusting, and the names Maxwell-Boltzmann and Laplace-Perrin attach to that prevented case.

6.1 The reference pair T_0 and P_0

The slice plus $dq = 0$ fixes the slope of the profile, not its intercept. In (17) and (23)–(26), T_0 and P_0 are any co-located physical pair on that adiabat: a kinetic temperature and the pressure at the same mass. The ideal-gas law pairs them for a fixed quantity of gas, and no radiative offset appears in the differential equations. Changing the chosen reference merely slides the labels along the same curve; it does not change the pairing. Absolute scale requires an independent energy constraint, for example the solar energy absorbed by the planet, the total thermal-plus-potential energy of the column, or one measured physical pair. The slice by itself does not set T_0 .

In particular, T_0 is not the effective radiating temperature T_{eff} defined by planetary energy balance,

$$\sigma T_{eff}^4 = \frac{F_S(1 - A)}{4} \quad (28)$$

Equation (28) is a global bookkeeping identity for the planetary ensemble. It does not identify a kinetic temperature at a chosen pressure, and it should not be used as the reference pair in (23) or (24). If the optical depth of the column changes, the altitude from which the last photons escape may move, but that is a change of where the existing profile is read out, not a demonstration that T_0 must be T_{eff} or that the adiabat itself must shift. Here it is enough that the unguarded hydrostatic mass determines the slope, that any physical pair on that slope may serve as (T_0, P_0) , and that the effective temperature of the planet is not a necessary fix to that pair. At each height the kinetic temperature and the pressure are already a physical pair through the ideal-gas law.

If, as a numerical convenience, one nevertheless takes the value $T_{eff} \approx 255 \text{ K}$ (about -18°C) as T_0 , the corresponding P_0 must still be a physical partner: the pressure at the height where the kinetic temperature is observed to be 255 K. On the mean environmental profile of about 6.5 K km^{-1} that height is about 5 km, not an arbitrary optical surface. For comparison, an exponential mass distribution with scale height $H = RT/g \approx 8.4 \text{ km}$ (using $R \approx 287 \text{ J kg}^{-1} \text{ K}^{-1}$ and a near-surface $T \approx 288 \text{ K}$) has half of its mass below $H \ln 2 \approx 5.8 \text{ km}$. That is the isothermal evaluation of the hydrostatic identity that half the column mass lies where the pressure is half the surface value. The same identity on the linear profile, which is (26) with the environmental lapse in place of g/c_p , puts that level at about 5.5 km, where the kinetic temperature is about 252 K. The 255 K height therefore sits a few hundred metres below the half-mass level and a few kelvin warmer, as one would expect if a linear temperature distribution on a bounded column places a representative kinetic temperature

near the middle of the mass rather than at either extremity. At 5 km the observed pressure is about 540 hPa. That is the P_0 which belongs with $T_0 = 255 \text{ K}$ on the mean environmental profile: hydrostatic balance together with the ideal-gas law along a lapse of 6.5 K km^{-1} gives $P/P_s = \left(\frac{T}{T_s}\right)^{\frac{g}{RT}} \approx 0.53$, or about 540 hPa from a surface pressure of 1013 hPa, in agreement with the standard atmosphere at 5 km.

7. Latent heat

When water vapour condenses (or freezes) as a mass cools, latent heat is released and the energy budget of Section 4 gains a term. For a mass m with water-vapour mass fraction X_w (the vapour portion of the water substance) and latent heat L ,

$$U = m c_p T + m g h + m X_w L \quad (29)$$

and $dU = 0$ gives [1,3]

$$c_p dT + g dh + L dX_w = 0 \quad (30)$$

$$\frac{dT}{dh} = -\frac{g}{c_p} - \frac{L}{c_p} \frac{dX_w}{dh} \quad (31)$$

On ascent, vapour decreases, so the second term reduces the cooling rate relative to Γ_d . A compact global-order estimate takes a mean near-surface water-vapour volume fraction of about 2.5 percent, converts to a mass fraction of about 1.58 percent, and interpolates condensation to vanishing vapour by 10 km. The implied heating is about 3.5 K km^{-1} , hence

$$\Gamma_s \approx 9.8 - 3.5 = 6.3 \text{ K km}^{-1} \quad (32)$$

against an observed global-mean environmental rate of about 6.5 K km^{-1} . Reversing the estimate from the observed mean implies a mean vapour volume fraction of about 2.3 percent, within the empirical range of 1 to 4 percent. The standard saturated-adiabatic formula [1,3] is the local, saturation-specific version of the same bookkeeping.

Moist static energy $c_p T + g h + L q$ is therefore the same story, not a different theory. Latent release appears when condensate forms in rising, cooling air, and a quiet saturated layer does not manufacture the wet adiabat merely by having U written as (29). The writing of (29) is licensed by the same $dq = 0$ as (14), now including the latent reservoir.

The sign of the water-vapour correction is relevant. If a radiative-reemission account that warms denser lower layers preferentially were an additive cause of the same gradient, water vapour, often called the strongest greenhouse gas, would be expected to steepen the lapse relative to the dry adiabat. Thermodynamics and the observations do the opposite: latent heat reduces $|\Gamma_d|$ from about 9.8 to about 6.5 K km^{-1} . The gradient is fixed by the energy closure, not steepened by an additive radiative differential on the same variable [6].

8. Objections

8.1 “The lapse is assumed, because $dq = 0$ is assumed”

Hydrostatics plus the ideal-gas law leave $T(h)$ open, so that an energy closure is required, and that much is agreed. The claim that (14) is therefore a confession that the lapse was put in by hand treats the constitutive statement “no heat bath” as if it were a free function $q(h)$ chosen to match the weather, which it is not. The unguarded hydrostatic mass of Section 2 has, by definition, no thermostat, and equation (14) is that definition written in the First Law. The rival definition is the heat bath, which produces (21), and mutual exclusivity forbids running both. Choosing (14) is choosing the system under study, not inserting g/c_p ; once the system is chosen, (16) is derived, not fitted.

8.2 “Molecular streams do not perform $P dV$ as parcels”

A single ballistic molecule does not “do $P dV$.” A fluid mass at defined pressure does, and condition N1 already places the problem in the fluid regime. Pressure is the momentum flux of the same molecules that constitute the streams, and hydrostatic balance is that momentum flux in equilibrium with weight. The energy of a fluid mass whose pressure changes is enthalpy. Mean free path of order $0.1 \mu\text{m}$ makes the gravitational increment over one flight negligible, and the next collision rebuilds a local Maxwellian, rebuilt at the enthalpy appropriate to the new height because the pressure has changed. The continuum limit of that process is (13). Demanding a visually identified parcel is an extra and unnecessary visualisation: the textbook parcel is a convenient name for a unit mass, and the slice is the same unit mass, viewed as streams through a plane of rest.

8.3 “Identifying $\frac{1}{2} v_0^2$ with $c_p T$ inserts g/c_p by hand”

Agreed, if that identification were the derivation, which it is not. The derivation is the First Law (11), hydrostatics (7), and the unguarded closure (14). The speed identification is a commentary that recovers (15) from (3) and (4), and it may be omitted without loss. Using c_v in the First Law would be equally “kinetic” and would give the wrong rate, which is precisely why the First Law at defined pressure uses c_p . That is not circular; it is the distinction between a rigid box and a hydrostatic fluid.

8.4 “If there is no bulk motion, there is no expansion work, and one does not get c_p ”

Vanishing bulk vertical velocity means (1), not vanishing J . Expansion work is performed whenever a unit mass changes pressure, and hydrostatics guarantees that pressure changes with height. Thermal crossings of the slice change height, and therefore change pressure and perform expansion work, with zero net mass flux. Bulk overturning is a sufficient condition for mixing a deep layer toward (17), but it is not a necessary condition for (16) to be the energy law of the slice. Confusing “no mean wind” with “no energy exchange on the slice” is the error.

8.5 “A laboratory cylinder never displays the lapse, or never goes isothermal”

A metre-scale cylinder cannot decide between the two accounts. The scale height is about 8 km, so that over a metre the dry adiabat is about 0.01 K and $m g h / k T \sim 10^{-4}$. The walls are a heat bath, and they impose one T . There is no 10 km insulating cylinder of dry air left alone for the conductive time L^2/κ , which at atmospheric scales is geological. Failure to observe -9.8 K km^{-1} from floor to ceiling is therefore not evidence that a gravitational atmosphere is isothermal; it is evidence that a thermostat dominates a small box. In a room one measures the walls.

The atmosphere is the experiment that has the vertical scale. Where the gas is left to gravity and enthalpy, the mixed troposphere occupies a (moist) adiabat, not one β . That is the empirical content of the unguarded hydrostatic energy account of Section 5.

8.6 “The atmosphere refutes a mandatory lapse: inversions, tropopause, stratosphere”

Nocturnal inversions, the tropopause, and a stratosphere that warms with height are collisional gases in gravity. They are also layers in which $dq \neq 0$. The ground radiates at night, ozone absorbs ultraviolet in the stratosphere, and the column becomes optically thin near the tropopause. Equation (13) then has a non-zero heat term and the profile leaves (16), which confirms the closure rather than refuting it. If $-g/c_p$ were claimed to hold in every layer regardless of local heating, the stratosphere would be a counterexample, but that is not the claim. The claim is that for an unguarded hydrostatic ideal gas with $dq = 0$, the lapse is mandatory. Where heat is added or removed, the energy law is (13), not (15).

8.7 “The dry adiabat is only the Schwarzschild boundary, not the unique static state”

The Schwarzschild criterion [1,2] states that a hydrostatic layer is convectively unstable if its environmental gradient is steeper than Γ_d (or Γ_s if saturated), and convection then mixes the layer toward the isentrope. That is the stability statement of the same energy law, not a rival theory: it says when a radiative or conductive profile cannot persist, and it does not say that Γ_d is optional as the energy equilibrium of an unguarded mass. Heating from below, or a radiative profile steeper than Γ_d , decides whether overturning breaks out. Overturning realises (17); it does not write g/c_p . Without that driving, radiation or walls may impose another slope, which is again $dq \neq 0$ or a heat bath. The dry adiabat remains the entropy-conserving displacement of a compressible ideal gas in hydrostatic pressure, which is a constraint on a process and the equilibrium of the process the unguarded slice undergoes.

9. Empirical content and physical necessities restated

The dry rate (18) is the standard handbook value. The moist estimate (32) matches the observed global-mean environmental lapse to within a few tenths of a kelvin per kilometre, using independently measured g , c_p , L , and a water-vapour fraction within the observed range. No residual in the mean

tropospheric gradient remains that must be filled by a second concurrent energy closure on the same variable [1-3,6].

Heating from below is the reason a planetary troposphere is mixed rather than left to a purely radiative profile, but it is not a term in (15). The equation does not mention it because the equation is the energy law of a hydrostatic mass, not a model of the solar beam. The solar beam decides whether the column is driven through the Schwarzschild boundary, while gravity and enthalpy decide what slope the mixed column occupies.

The physical necessities of Section 2 may now be read as the conditions under which (16) is mandatory: a collisional hydrostatic ideal gas, not coupled to a thermostat, with vanishing net mass flux through each horizontal surface and non-vanishing thermal streams. Drop the thermostat and keep the rest, and the lapse is the energy equilibrium; restore the thermostat, and the isothermal barometer is the energy equilibrium. Those are the two complete accounts, and they are mutually exclusive.

10. Conclusions

1. Mass, momentum, and energy on an infinitesimal horizontal slice are the zeroth, first, and second moments of the same construction: mass gives equal and opposite thermal streams, momentum gives hydrostatic balance, and energy, at locally defined pressure, gives $dq = c_p dT + g dh$.
2. For an unguarded hydrostatic ideal gas, $dq = 0$ is the constitutive statement that there is no heat bath. The dry adiabat $\frac{dT}{dh} = -g/c_p$ follows at once, together with Poisson's pressure law. The same pair is recovered from the First Law of a reversible adiabatic parcel plus hydrostatics. That is one closure, written twice.
3. The isothermal barometer is the equilibrium of a thermostatted column, equivalently the canonical distribution in an external potential. It is a complete account of a different system, and it is not a state the unguarded hydrostatic atmosphere occupies. Equation (10) is the formal integral of hydrostatics if temperature is forced constant, a forcing the energy moment of the slice does not grant.
4. The two closures are mutually exclusive on dT/dh . The overlooked exclusivity is not Maxwell versus a collisionless cartoon; it is enthalpy of an unguarded mass versus molecular energy in a heat bath. Assigning the latter to "the equilibrium of a real gas in gravity" is the false step.
5. Bulk convection and heating from below are not premises of (16). They are, in a planetary atmosphere, the means by which a deep layer is driven through the Schwarzschild boundary and mixed onto the isentrope. They do not write g/c_p . Inversions and the stratosphere are $dq \neq 0$, and laboratory cylinders are thermostats; neither class of observation refutes the energy equilibrium of the unguarded slice.

6. The quantity $U = m c_p T + m g h$ is therefore not too simple: it is the energy of the system under study, and its differential vanishing is the dry adiabat, or with latent heat included the moist adiabat. The troposphere occupies that equilibrium. The isothermal column is the prevented case, namely what would remain if walls, or a bath, or an infinite wait for conduction, were allowed to undo every enthalpy-gravity exchange the streams have just made.

The reality of mutual exclusivity here is the reality of non-contradiction applied to two complete energy accounts of one gradient. Empirically successful tropospheric thermodynamics is not an optional narrative beside a thermostatted barometer; it is the accounting that leaves no room for that barometer on the same variable, in the same unguarded gas. Science progresses when that room is acknowledged to be full.

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Appendix A. Parcel derivation of the dry adiabat

For completeness, the standard parcel route is written without abbreviation. The First Law per unit mass for a reversible process in an ideal gas is

$$dq = c_p dT - dP / \rho \quad (\text{A1})$$

Adiabatic means $dq = 0$. Hydrostatic balance is $dP = -\rho g dh$. Substitution gives

$$c_p dT + g dh = 0 \quad (\text{A2})$$

$$\frac{dT}{dh} = -g/c_p \quad (\text{A3})$$

which is (16). The slice derivation of Section 4 is this same argument, with the unit mass identified as the thermal streams through a plane of rest rather than as a labelled balloon, and the physics is identical.

Appendix B. The energy of a rigid box

The specific internal energy of an ideal gas is $c_v T$. In a rigid closed volume the total energy available to a gravitational Hamiltonian is

$$E = \int \rho(c_v T + g h) dV \quad (\text{B1})$$

Maximising the entropy of a classical gas at fixed E and fixed mass yields one β and (21). The extra $R T$ that converts c_v into c_p is absent because a rigid box does not permit the mass to change volume against a neighbouring pressure. A hydrostatic atmosphere is not a rigid box: the pressure of each mass is the weight of the fluid above it, and that pressure changes with height. The First Law at defined pressure therefore uses c_p . Conserving (B1) and conserving $c_p T + g h$ are the two closures of Section 5, and they cannot both be the equilibrium of the same system.